5.8 Notes and Examples

Name: Block: Seat:

Inverse Trig Integration

1. Warm Up: Complete the square:

   (a) $x^2 + 6x + \boxed{9} = (x + \underline{3})^2$

   (b) $x^2 - 8x + 12 = (x - \underline{4})^2 - \underline{4}$

2. Warm up: find $f'(x)$

   Chain

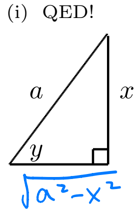
   (a) $f(x) = \arcsin \frac{x}{5} = \frac{1}{5} \cdot \frac{1}{\sqrt{1-\frac{x}{5}}} = \frac{1}{\sqrt{25}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} = \frac{1}{\sqrt{25-x^2}}$

   (b) $f(x) = \frac{1}{5} \arctan \frac{x}{5} = \frac{1}{5} \cdot \frac{1}{1+\left(\frac{x}{5}\right)^2} \cdot \frac{1}{5} = \frac{1}{25\left(1+\frac{x^2}{25}\right)} = \frac{1}{25+x^2}$

   Coeff Chain Rule

3. Deriving the Derivative of $y = \arcsin \frac{x}{a}$

   (a) Draw a right triangle with hypotenuse a, acute angle y and opposite leg with length x.
   (b) Use the Pythagorean Theorem to find the length of the adjacent leg.
   (c) Start with $y = \arcsin \frac{x}{a}$ (note how this true from our drawing)
   (d) Take the sine of both sides (note how this can also be verified from the drawing SOH-CAH-TOA)
   (e) Implicitly differentiate both sides with respect to x (Remember the chain rule)
   (f) Substitute $\cos y$ with "adjacent over hypotenuse" from the drawing.
   (g) Multiply both sides by a
   (h) Solve for $\frac{dy}{dx}$
   (i) QED!



$y = \arcsin \frac{x}{a}$

$\sin y = \frac{x}{a}$

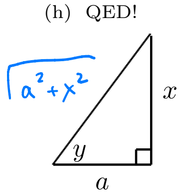
$\cos y \frac{dy}{dx} = \frac{1}{a}$

$\frac{\sqrt{a^2-x^2}}{a} \frac{dy}{dx} = \frac{1}{a}$

$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{a}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}}$

4. Deriving the Derivative of $y = \arctan \frac{x}{a}$

   (a) Draw a right triangle with adjacent leg a, acute angle y and opposite leg with length x.
   (b) Use the Pythagorean Theorem to find the length of the hypotenuse.
   (c) Start with $y = \arctan \frac{x}{a}$ (note how this true from our drawing)
   (d) Take the tangent of both sides (note how this can also be verified from the drawing SOH-CAH-TOA)
   (e) Implicitly differentiate both sides with respect to x (Remember the chain rule)
   (f) Substitute $\sec^2 y$ with "hypotenuse over adjacent squared" from the drawing
   (g) Solve for $\frac{dy}{dx}$
   (h) QED!



$y = \arctan \frac{x}{a}$

$\tan y = \frac{x}{a}$

$\sec^2 y \frac{dy}{dx} = \frac{1}{a}$

$\left(\frac{\sqrt{a^2+x^2}}{a}\right)^2 \frac{dy}{dx} = \frac{1}{a}$

$\frac{a^2+x^2}{a^2} \frac{dy}{dx} = \frac{1}{a}$

$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{a^2}{a^2+x^2}$

$\frac{dy}{dx} = \frac{a}{a^2+x^2}$

The Theorems

If u is a function of x and a is a constant:

$$\boxed{\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C}$$

$$\boxed{\int \frac{a}{a^2+u^2}\,du = \arctan\frac{u}{a} + C}$$

so that

$$\boxed{\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C}$$

1. Examples

(a) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ $= \int \frac{1}{\sqrt{a^2-u^2}}\,du$

$= \arcsin\frac{u}{a} + C$

$$\boxed{\begin{aligned} a^2 &= 4 \\ a &= 2 \\ u &= x \\ du &= dx \end{aligned}}$$

$= \arcsin\frac{x}{2} + C$

(b) $\int \frac{1}{9+x^2}\,dx$ $= \int \frac{1}{3^2+x^2}\,dx =$

$$\boxed{\begin{aligned} a^2 &= 9 \\ a &= 3 \\ u &= x \\ du &= dx \end{aligned}}$$

$= \int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$

$\frac{1}{3}\arctan\frac{x}{3} + C$

(c) $\int \frac{1}{\sqrt{3-x^2}}\,dx$

$a^2 = 3$
$a = \sqrt{3}$
$u = x$
$du = dx$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du$

$= \arcsin \frac{u}{a} + C$

$= \arcsin \frac{x}{\sqrt{3}} + C$

(d) $\int \frac{1}{4+25x^2}\,dx = \int \frac{1}{(2)^2+(5x)^2}\,dx$ or $\frac{1}{5}\int \frac{1}{a^2+u^2}\,du$

$a = 2$
$u^2 = 25x^2$
$u = 5x$
$du = 5\cancel{dx}$
$\frac{1}{5}du = dx$

$\frac{1}{5}\cdot\frac{1}{a}\arctan\frac{u}{a} + C$

$= \frac{1}{5}\cdot\frac{1}{2}\arctan\frac{5x}{2} + C$

$= \frac{1}{10}\arctan\frac{5x}{2} + C$

(e) $\int \frac{1}{2+9x^2}\,dx = \frac{1}{3}\int \frac{1}{(a)^2+u^2}\,du = \frac{1}{3}\cdot\frac{1}{a}\arctan\frac{u}{a} + C$

$a = \sqrt{2}$
$u = 3x$
$du = 3\,dx$
$\frac{1}{3}du = dx$

$= \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\arctan\frac{3x}{\sqrt{2}} + C$

(f) $\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{1/2} \frac{1}{\sqrt{a^2-u^2}}\,du = \frac{1}{3}\arcsin\frac{u}{a}\Big|_0^{1/2}$

$a = 1$
$u = 3x$
$du = 3\,dx$
$\frac{1}{3}du = dx$
$u(0) = 0$
$u(1/6) = 1/2$

$= \frac{1}{3}\left[\arcsin\frac{1}{2} - \arcsin 0\right]$

$= \frac{1}{3}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{18}$

2. **Completing the Square:** Completing the square is a technique you learned in Algebra 2 for solving quadratics. It is used in Calculus to set up integrals that require the inverse trig rules. Recall that in order to complete the square, we take half of the coefficient in front of x , square it, and add it to the problem, and then factor. In order to not change the problem, I need to subtract this term as well, so that it seems like I am "adding zero"

   (a) Write $x^2 - 4x + 7$ in the form $(x-h)^2 + k$ $= (x^2 - 4x + 2^2) - 2^2 + 7)$

   $(x-2)^2 + 3$

   (b) $\int \frac{1}{x^2 - 4x + 7}\, dx = \int \frac{1}{(x-2)^2+3} dx$

   $\boxed{a = \sqrt{3} \quad u = x-2}$

   $\int \frac{1}{a^2+u^2} du$

   $= \frac{1}{a} \arctan \frac{u}{a} + C$

   $= \frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C$

   (c) Write $x^2 - 6x + 13$ in the form $(x-h)^2 + k$ $= (x^2 - 6x + 3^2) - 3^2 + 13$

   $= (x-3)^2 + 4$

   (d) $\int \frac{1}{x^2 - 6x + 13}\, dx = \int \frac{1}{4 + (x-3)^2} dx$

   $\boxed{a = 2 \quad u = (x-3)}$

   $\int \frac{1}{a^2+u^2} du$

   $= \frac{1}{a} \arctan \frac{u}{a} + C$

   $= \frac{1}{2} \arctan \frac{x-3}{2} + C$

   (e) Write $3x - x^2$ in the form $k - (x-h)^2$: $-\left(x^2 - 3x + \left(\frac{3}{2}\right)^2\right) + \left(\frac{3}{2}\right)^2$

   $-\left(x-\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2$

   $\left(\frac{3}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2$

   (f) $\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx = \int_{3/2}^{9/4} \frac{1}{\left(\frac{3}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} dx$

   $\int_0^{3/4} \frac{1}{\sqrt{a^2-u^2}} du$

   $\boxed{a = \frac{3}{2} \quad u = \left(x-\frac{3}{2}\right) \quad u\left(\frac{3}{2}\right) = 0 \quad u\left(\frac{9}{4}\right) = \frac{9}{4} - \frac{6}{4} = \frac{3}{4}}$

   $= \arcsin \frac{u}{a} \Big|_0^{3/4}$

   $= \arcsin\left(\frac{3}{4} \cdot \frac{2}{3}\right) - \arcsin(0)$

   $= \arcsin \frac{1}{2} - 0$

   $= \frac{\pi}{6}$